THE BERGMAN KERNEL OF THE SYMMETRIZED POLYDISC IN HIGHER DIMENSIONS HAS ZEROS

NIKOLAI NIKOLOV AND WŁODZIMIERZ ZWONEK

Abstract. We prove that the Bergman kernel of the symmetrized polydisc in dimension greater than two has zeros.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ and $\pi_n = (\pi_{n,1}, \dots, \pi_{n,n}) : \mathbb{C}^n \to \mathbb{C}^n$ be defined as follows:

$$\pi_{n,k}(\lambda) = \sum_{1 \le j_1 < \dots < j_k \le n} \lambda_{j_1} \dots, \lambda_{j_k}, \quad 1 \le k \le n.$$

The set $\mathbb{G}_n = \pi_n(\mathbb{D}^n)$ is called the symmetrized polydisc. The symmetrized bidisc $\mathbb{G}_2$ is the first example of a bounded pseudoconvex (even hyperconvex) domain that cannot be exhausted by domains biholomorphic to convex domains and on which the Carathéodory and Kobayashi distances coincide (see [3], [4] and [1], see also [5]). Note that $\mathbb{G}_n$, $n \ge 3$, cannot be exhausted by domains biholomorphic to convex domains, too (see [7]). It is, however, not known whether the Carathéodory and Kobayashi distances coincide on $\mathbb{G}_n$, $n \ge 3$.

In [6], the following explicit formula for the Bergman kernel $K_{\mathbb{G}_n}$ of $\mathbb{G}_n$ has been found:

(1)
$$K_{\mathbb{G}_n}(\pi_n(\lambda), \pi_n(\mu)) = \frac{\det[(1 - \lambda_j \overline{\mu}_k)^{-2}]_{1 \le j,k \le n}}{\pi^n \prod_{1 \le j < k \le n}[(\lambda_j - \lambda_k)(\overline{\mu}_j - \overline{\mu}_k)]}, \quad \lambda, \mu \in \mathbb{D}^n.$$

Observe that although the right-hand side of (1) is not formally defined on the whole $\mathbb{G}_n \times \mathbb{G}_n$, it extends smoothly on this set. The formula (1) easily implies that $\mathbb{G}_2$ is a Lu Qi-Keng domain (see [6]), i.e. $K_{\mathbb{G}_2}$ has no zeros on $\mathbb{G}_2 \times \mathbb{G}_2$ – for the comprehensive information on the Lu Qi-Keng problem see e.g. [2]. Then the following natural question

2000 *Mathematics Subject Classification.* 32A25.

Key words and phrases. Bergman kernel, Lu Qi-Keng domain, symmetrized polydisc.

This note was prepared during the stay of the first named author at the Erwin Schrödinger Institute, Vienna in September, 2005 and at the Jagiellonian University, Kraków in November, 2005. He thanks both institutions for their hospitality. The work is a part of the Research Grant No.1 PO3A 005 28 which is supported by public means in the programme promoting science in Poland in the years 2005-2008.

has been posed in [6] (see also [5]): Does $K_{\mathbb{G}_n}$ have zeros for $n \geq 3$? The aim of this note is to give a positive answer to the above question thus providing an example of a proper image of the polydisc $\mathbb{D}^n$, $n \geq 3$, which is not a Lu Qi-Keng domain.

Theorem 1. *$K_{\mathbb{G}_n}$ has zeros for any $n \geq 3$.*

Proof. We shall proceed by induction on $n \geq 3$ showing that:

$(*)$ there are points $\lambda, \mu \in \mathbb{D}^n$, both with pairwise different coordinates, such that

$$\Delta_n(\lambda, \mu) := \det[(1 - \lambda_j \overline{\mu}_k)^{-2}]_{1 \leq j,k \leq n} = 0$$

and $f_n := \Delta_n(\cdot, \lambda_2, \dots, \lambda_n, \mu_1, \dots, \mu_n) \not\equiv 0$.

The case $n = 3$. We have the following formula (see Appendix A):

$$K_{\mathbb{G}_3}(\pi_3(\lambda_1, \lambda_2, \lambda_3), \pi_3(\mu_1, \mu_2, 0)) = \frac{a(\nu)z^2 - b(\nu)z + 2c(\nu)}{\pi^3 \prod_{1 \leq j \leq 3, 1 \leq k \leq 2}(1 - \lambda_j \overline{\mu}_k)^2}, \tag{2}$$

where $z = \dfrac{\overline{\mu}_2}{\overline{\mu}_1}$ $(\mu_1 \neq 0)$, $\nu_j = \lambda_j \overline{\mu}_1$, $j = 1, 2, 3$, and

$$a(\nu) = \pi_{3,2}(\nu)(2 - \pi_{3,1}(\nu)) + \pi_{3,3}(\nu)(2\pi_{3,1}(\nu) - 3),$$
$$b(\nu) = (\pi_{3,1}(\nu) - 2)(\pi_{3,2}(\nu) - 2\pi_{3,1}(\nu) + 3) + 3(\pi_{3,3}(\nu) - \pi_{3,1}(\nu) + 2),$$
$$c(\nu) = \pi_{3,2}(\nu) - 2\pi_{3,1}(\nu) + 3.$$

For the fixed point $\nu_0 = (e^{i\pi/6}, e^{i\pi/3}, e^{-i\pi/6})$[1] the number

$$z_0 = e^{-i\pi/4} \frac{6 - 3\sqrt{3} - \sqrt{40\sqrt{3} - 69}}{\sqrt{2}(3\sqrt{3} - 5)}$$

satisfies the equality $a(\nu_0)z_0^2 - b(\nu_0)z_0 + 2c(\nu_0) = 0$ (see Appendix B). Since $z_0 \in \mathbb{D}$, it follows that for any $\nu \in \mathbb{D}^3$, close to ν_0, there is a $z \in \mathbb{D}$, close to z_0, such that $a(\nu)z^2 - b(\nu)z + 2c(\nu) = 0$. Then choosing $\mu_1 \in \mathbb{D}$ with $|\mu_1| > |\nu_1|, |\nu_2|, |\nu_3|$ we get points $\lambda, \mu \in \mathbb{D}^3$, both with pairwise different coordinates such that $\Delta_3(\lambda, \mu) = 0$.

To see that $f_3 \not\equiv 0$ assume the contrary. Then $f_3(0) = f_3'(0) = f_3''(0) = 0$, i.e.

$$\det \begin{bmatrix} \overline{\mu}_1^j & \overline{\mu}_2^j & \overline{\mu}_3^j \\ (1 - \lambda_2 \overline{\mu}_1)^{-2} & (1 - \lambda_2 \overline{\mu}_2)^{-2} & (1 - \lambda_2 \overline{\mu}_2)^{-2} \\ (1 - \lambda_3 \overline{\mu}_1)^{-2} & (1 - \lambda_3 \overline{\mu}_2)^{-2} & (1 - \lambda_3 \overline{\mu}_3)^{-2} \end{bmatrix} = 0$$

for $j = 0, 1, 2$. Since μ_1, μ_2, μ_3 are pairwise different, the vectors $(1, 1, 1)$, (μ_1, μ_2, μ_3) and $(\mu_1^2, \mu_2^2, \mu_3^2)$ are linearly independent. It follows that the vectors in the second and the third lines of the above determinant are

[1]We thank Pencho Marinov for his computer programme which shortened the calculations needed for finding good candidates for zeros of $K_{\mathbb{G}_3}$.

linearly dependent. In particular, $K_{\mathbb{G}_2}(\pi_2(\lambda_2,\lambda_3),\pi_2(\mu_2,\mu_3)) = 0$, a contradiction.

The induction step. Assume that $(*)$ holds for some $n \geq 3$. We shall choose numbers $\tilde\lambda_1$ and $\tilde\lambda_{n+1}$ in $\mathbb{D}$, close to λ_1 and 1, respectively (which will provide pairwise different coordinates of the new points), such that

$$g_{n+1}(\tilde\lambda_1,\tilde\lambda_{n+1}) := \Delta_{n+1}(\tilde\lambda_1,\lambda_2,\dots,\lambda_n,\tilde\lambda_{n+1},\mu_1,\dots,\mu_n,\tilde\lambda_{n+1}) = 0$$

and $g_{n+1}(\cdot,\lambda_{n+1}) \not\equiv 0$. Note that

$$g_{n+1}(\tilde\lambda_1,\tilde\lambda_{n+1}) = \frac{f_n(\tilde\lambda_1)}{(1-|\tilde\lambda_{n+1}|^2)^2} + h_n(\tilde\lambda_1,\tilde\lambda_{n+1}),$$

where h_n is a continuous function on $\mathbb{D}\times\overline{\mathbb{D}}$. Since $f_n \not\equiv 0$ is a holomorphic function, for any small $r > 0$ the number λ_1 is the only zero of f_n in the closed disc $D \subset \mathbb{D}$ with center at λ_1 and radius r. Then $m := \dfrac{\min_{\partial D}|f_n|}{\max_{\partial D\times\overline{\mathbb{D}}}|h_n|} > 0$. Hence $|f_n| > (1-|\tilde\lambda_{n+1}|^2)^2|h_n(\cdot,\tilde\lambda_{n+1})|$ on ∂D if $1-|\tilde\lambda_{n+1}|^2 < \sqrt{m}$. Fix such a $\tilde\lambda_{n+1}$. Since $h_n(\cdot,\tilde\lambda_{n+1})$ is a holomorphic function on $\mathbb{D}$, the Rouché theorem implies that $g_{n+1}(\cdot,\tilde\lambda_{n+1})$ has finitely many zeros in D (in particular, $g_{n+1}(\cdot,\tilde\lambda_{n+1}) \not\equiv 0$), which completes the proof.

$\square$

Remark. The above proof shows that if $n \geq 4$, then there are points (λ,ν), close to the diagonal of $\mathbb{D}^n\times\mathbb{D}^n$ in the following sense: $\lambda_j = \mu_j > 0$ for $j = 4,\dots,n$ such that $K_{\mathbb{G}_n}(\pi_n(\lambda),\pi_n(\mu)) = 0$. On the other hand, it can be shown that $K_{\mathbb{G}_3}(\pi_3(\lambda),\pi_3(\mu)) \neq 0$ if $\lambda_3 = \mu_3$.

Appendix A. By (1), one has that

(3)

$$\pi^3(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)\overline{\mu}_1\overline{\mu}_2(\overline{\mu}_1-\overline{\mu}_2)K_{\mathbb{G}_3}(\pi_3(\lambda_1,\lambda_1,\lambda_3),\pi_3(\mu_1,\mu_2,0))$$

$$= \det\begin{bmatrix}(1-\nu_1)^{-2} & (1-z\nu_1)^{-2} & 1\\ (1-\nu_2)^{-2} & (1-z\nu_2)^{-2} & 1\\ (1-\nu_3)^{-2} & (1-z\nu_3)^{-2} & 1\end{bmatrix}$$

$$= \det\begin{bmatrix}(1-\nu_1)^{-2}-(1-\nu_3)^{-2} & (1-z\nu_1)^{-2}-(1-z\nu_3)^{-2}\\ (1-\nu_2)^{-2}-(1-\nu_3)^{-2} & (1-z\nu_2)^{-2}-(1-z\nu_3)^{-2}\end{bmatrix}$$

$$= \frac{(\nu_1-\nu_3)(\nu_2-\nu_3)z}{(1-\nu_3)^2(1-z\nu_3)^2}\det\begin{bmatrix}\dfrac{\nu_1+\nu_3-2}{(1-\nu_1)^2} & \dfrac{z\nu_1+z\nu_3-2}{(1-z\nu_1)^2}\\ \dfrac{\nu_2+\nu_3-2}{(1-\nu_2)^2} & \dfrac{z\nu_2+z\nu_3-2}{(1-z\nu_2)^2}\end{bmatrix}$$

$$= \frac{(\nu_1-\nu_3)(\nu_2-\nu_3)z}{\prod_{1\le j\le 3, 1\le k\le 2}(1-\lambda_j\overline{\mu}_k)^2}\Big((\nu_1+\nu_3-2)(z\nu_2+z\nu_3-2)(1-z\nu_1)^2(1-\nu_2)^2$$

$$-(\nu_2+\nu_3-2)(z\nu_1+z\nu_3-2)(1-\nu_1)^2(1-z\nu_2)^2\Big) \tag{4}$$

$$= \frac{(\nu_1-\nu_3)(\nu_2-\nu_3)z(z-1)(A(\nu)z^2-B(\nu)z+2C(\nu))}{\prod_{1\le j\le 3, 1\le k\le 2}(1-\lambda_j\overline{\mu}_k)^2}. \tag{5}$$

To find $A(\nu)$, $B(\nu)$ and $C(\nu)$, we shall use that the coefficients of z^3, z^0 and z in the large brackets in (4) are equal to

$$A(\nu) = (\nu_1+\nu_3-2)(\nu_2+\nu_3)\nu_1^2(1-\nu_2)^2-(\nu_2+\nu_3-2)(\nu_1+\nu_3)\nu_2^2(1-\nu_1)^2,$$

$$-2C(\nu) = 2(\nu_2+\nu_3-2)(1-\nu_1)^2-2(\nu_1+\nu_3-2)(1-\nu_2)^2 \text{ and}$$

$$B(\nu)+2C(\nu) = (\nu_1+\nu_3-2)(\nu_2+\nu_3+4\nu_1)(1-\nu_2)^2$$

$$-(\nu_2+\nu_3-2)(\nu_1+\nu_3+4\nu_2)(1-\nu_1)^2,$$

respectively. Calculations lead to the formulas

$$A(\nu) = (\nu_2-\nu_1)(\pi_{3,2}(\nu)(2-\pi_{3,1}(\nu))+\pi_{3,3}(\nu)(2\pi_{3,1}(\nu)-3)),$$

$$C(\nu) = (\nu_2-\nu_1)(\pi_{3,2}(\nu)-2\pi_{3,1}(\nu)+3),$$

$$B(\nu) = (\nu_2-\nu_1)((\pi_{3,1}(\nu)-2)(\pi_{3,2}(\nu)-2\pi_{3,1}(\nu)+3)+3(\pi_{3,3}(\nu)-\pi_{3,1}(\nu)+2)).$$

To get (2), it remains to substitute these formulas in (5) and then to compare (5) and (3).

Appendix B. Since

$$\pi_{3,1}(\nu_0) = \frac{1+2\sqrt{3}+i\sqrt{3}}{2},\ \pi_{3,2}(\nu_0) = \frac{2+\sqrt{3}+i3}{2},\ \pi_{3,3}(\nu_0) = e^{i\pi/3},$$

the formulas for $a(\nu)$, $b(\nu)$ and $c(\nu)$ lead to

$$a(\nu_0) = (3\sqrt{3}-5)e^{i\pi/3},\ b(\nu_0) = (6\sqrt{2}-3\sqrt{6})e^{i\pi/12},\ c(\nu_0) = (2\sqrt{3}-3)e^{-i\pi/6}.$$

Then for $z = e^{-i\pi/4}x$ one has $e^{i\pi/6}(a(\nu_0)z^2-b(\nu_0)z+2c(\nu_0))$

$$= (3\sqrt{3}-5)x^2+(3\sqrt{6}-6\sqrt{2})x+4\sqrt{3}-6 =: p(x).$$

The zeros of the polynomial p are equal to $\dfrac{6-3\sqrt{3}\pm\sqrt{40\sqrt{3}-69}}{\sqrt{2}(3\sqrt{3}-5)}$.
Note that the smaller one lies in $(0,1)$, since $p(0) > 0 > p(1)$.

References

[1] J. Agler, N. J. Young, *The hyperbolic geometry of the symmetrized bidisc*, J. Geom. Anal. 14 (2004), 375–403.

[2] H. P. Boas, *Lu Qi-Keng's problem*, J. Korean Math. Soc. 37 (2000), 253–267.

[3] C. Costara, *Dissertation*, Universite Laval (2004).

[4] C. Costara, *The symmetrized bidisc and Lempert's theorem*, Bull. London Math. Soc. 36 (2004), 656–662.

[5] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis–revisited*, Diss. Math. 430 (2005), 1–192.

[6] A. Edigarian, W. Zwonek, *Geometry of the symmetrized polydisc*, Arch. Math. 84 (2005), 364-374.

[7] N. Nikolov, *The symmetrized polydisc cannot be exhausted by domains biholomorphic to convex domains*, preprint, 2005 (arXiv:math.CV/0507190).

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria

E-mail address: nik@math.bas.bg

Instytut Matematyki, Uniwersytet Jagielloński, Reymonta 4, 30-059 Kraków, Poland

E-mail address: Wlodzimierz.Zwonek@im.uj.edu.pl